Marian Smoluchowski and the Theory of Probabilities in Physics

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A brief account of the life of the Polish physicist Marian Smoluchowski, with some observations on the implications of his scientific work.

It is forty years since the death of Marian Smoluchowski, a great Polish theoretical physicist (see Fig. 1). A contemporary of Einstein (older by a few years), he died during the First World War in Cracow in September, 1917. A book published in Poland last year by A. Teske[1] describes his life and his work.

The work of Smoluchowski in the theory of fluctuations and kinetic theory of gases, especially in the theory of Brownian motions, is too well known to physicists to need renewed presentation. It is interesting, though, to survey these pioneering contributions from the perspective of a few decades and observe how the ideas of the theory of probabilities influenced the development of the kinetic or "particle" point of view in theoretical physics. Smoluchowski, together with and independently of Einstein, elaborated and carried forward the ideas of Maxwell and Boltzmann. Some of his fundamental contributions concerned the role of statistical fluctuations in phenomena involving assemblies of particles, and confirmed their importance in explaining phenomena like the Brownian motion and opalescence. One might say that, before him, most studies were concerned with the thermodynamic variables representing the first means or expected values of the relevant random variables. Because it went further into an examination of the deviations and second moments, the work of Smoluchowski gave further confirmation to the reality of a kinetic picture of matter.

It is instructive to consider the increasing role of the probabilistic approach during the forty years that followed his work. For one thing, quantum theory generalized this approach, extended it, and made it even more basic. It is true that what was called an assembly of elementary particles now forms a more general, if less concrete, picture. Sometimes, in considering nuclear phenomena, one pictures a collection of a considerable number of "virtual" particles or events. The statistical approach like the one, for example, used by Fermi for collision processes in meson production is often very useful even when the number of particles involved is rather small! Here again, the study of fluctuations or deviations from first means is especially important, since the number of the particles is large, though not enormously so. Moreover, in the more classical fields of physics,

Fig. 1. Marian Smoluchowski.

[1] Armin Teske, *Marian Smoluchowski, His Life and Work* (Polish State Scientific Publishing House, Krakow, Poland, 1955).

for instance in hydrodynamics,[2] the ideas which originate historically in thermodynamics (that is statistical treatment) find ever wider applications—to mention only the theory of turbulence in this connection!

It is interesting, even today, to reread some general speculations of Smoluchowski on the idea of chance and the origin of principles of the theory of probabilities in physics, published posthumously in his article in *Naturwissenschaften* in 1918, and then follow the subsequent development of ideas in this direction!

A book published in Poland last year entitled *Selection of Philosophical Writings of Smoluchowski*, contains a number of his more general articles, including some celebrated ones: on the second law of thermodynamics (Gültigkeitsgrenzen des zweiten Hauptsatzes der Wärmetheorie), and a paper in the Boltzmann-Festschrift, "Über die Unregelmässigkeiten in der Verteilung von Gasmolekülen und deren Einfluss auf Entropie und Zustandsgleichung," Another important paper is Molekulartheoretische Studien über Umkehrung thermodynamisch irreversibler Vorgänge und über Wiederkehr abnormaler Zustände.[3] It also contains several lectures; one is on the importance of exact sciences in general education given in 1917—still a vital topic—full of remarks relevant even to the present discussions in this country on the relative role of traditional classical education *versus* a more technical one. There is a beautiful article on "The Subject, Problems, and Methods in Physics," taken from a publication, *A Guide for Self-Study in Science*, published in Poland in 1917.

Teske's biography gives a nice account of the scientific atmosphere and life in European universities at the turn of the last century by following the travels of Smoluchowski in Germany, Italy, France, and England and his work in Vienna, Lwow, and Cracow. It is interesting and perhaps melancholy to compare the degree of mutual influence of different centers then, with the degree of scientific cooperation between central and eastern Europe on one hand and the West on the other, at the present time.

Smoluchowski's studies and his post-doctoral travels give a very vivid impression of the genetic development of many of the ideas of modern physics in their germinating stages. In physics, this period was, perhaps, one of a calm before the storm of the full impact of the theory of relativity and quantum theory. One feels inklings of an impending revolution in the foundations of physics, and since the present time, perhaps, gives a similar impression, it may be instructive to follow in retrospect the appearance at that time of the signs and omens of change to come.

Smoluchowski was born in Vienna in 1872, the son of a Cracow lawyer who was later secretary of the Chancellery at the Court of the Emperor Francis Joseph. Young Smoluchowski had the benefit of exceptionally favorable conditions in his childhood and youth. His mother, very well educated, created at home an artistic and intellectual atmosphere. Vienna, the capital, was a brilliant center of art and science in its day. Smoluchowski studied at the famous Theresianum, the best "gymnasium," attended by sons of the nobility and high government officials.

In his high school years, young Smoluchowski showed interest in science, especially astronomy, and, thanks to his teacher, A. Hofler (later a professor of the University of Vienna), became interested in problems of physics. After graduating from the gymnasium, he entered the University of Vienna where he elected physics and mathematics as his fields and attended the lectures of Boltzmann. In all his work he kept a predilection for stressing equally theoretical and experimental aspects. His admiration for Boltzmann continued throughout the rest of his life. In fact, Smoluchowski was called later "der geistige Nachfolger Boltzmanns." Among his favorite professors were Exner, Mach, Lechner, and the philosopher, Jagic. He published his first scientific paper during the second year of his university studies. He graduated with the highest honors and received his doctorate "Sub Auspiciis Imperatoris." (This might amuse the modern reader: the most brilliant graduates at Austrian universities were distinguished by a

[2] Smoluchowski made important contributions to hydrodynamics. Cf. his paper in the Bull. Acad. Sci. Cracovie (1904), pp. 371–384.

[3] Sitzungsberichte der Akademie der Wissenchaften in Wien; mathem.-natur Klasse, Abt. IIa, 124. Bd; 5 Heft, 339–368 (1915).

special reward given in the name of the Emperor, including a diamond ring.)

Smoluchowski spent the year 1895–1896 in Paris working with Lipmann. He expressed in his letters a great appreciation of the intellectual atmosphere of Paris but also some disappointment at the lack of contact between the professors and the younger scientists. Next year, before going to Glasgow, he spent some time vacationing in Eastbourne. The summer was an occasion for mountaineering in Scotland. This activity started in childhood as a real passion and continued as a hobby in later years.[4]

The sojourn in Glasgow made a very strong impression on young Smoluchowski, and the influence was visible in his later life; many of his future attitudes and ideas had their source there, some of them due to Lord Kelvin. Smoluchowski was greatly impressed by the spirit of the laboratory to which visitors came from all over the world. For years, Smoluchowski remembered the high level of experimental and theoretical work there, in seeming contradiction to the very modest buildings and inadequate facilities. In a letter to his brother dated January, 1897, he wrote: "Lord Kelvin seems always greatly excited. Every day some different problem seems most interesting to him and he forgets everything else. He is impatient and cannot perform the experiments himself but could keep busy with his ideas a whole army of experimental physicists."

The next year, he spent several months in Berlin working in the laboratory of Professor Warburg in Charlottenburg near Berlin. Returning to Vienna, he became a lecturer (Dozent) and started lectures in mathematics and physics (quantum theory). He received an offer of a professorship for three years in Bombay, but after considerable hesitation, he decided not to accept it, and instead, in 1899, became a lecturer in Lwow. The next year, at 28, he was promoted to a professorship of theoretical physics. This made him the youngest professor in Austria.

In the Spring of 1901, he visited Glasgow for the 450th anniversary of the foundation of the university and received an honorary doctorate there. His marriage also took place in 1901.[5]

The fourteen years of his stay in Lwow formed a period of increasing activity and ever-expanding theoretical work. His teaching was extremely successful, and inspired a whole generation of young physicists and mathematicians. Despite his popularity with students and colleagues, however, he considered teaching to be a secondary aim. He continued his research and worked feverishly at publishing a whole series of important papers. The scientific atmosphere at the university was, before his coming, rather poor, at least in his field; it was mainly through correspondence that Smoluchowski kept contact with other physicists of the world.

From August 1905, to April 1906, taking a leave of absence, he went to Cambridge and worked at the famous Cavendish Laboratory. Among the people in his own fields of interest with whom Smoluchowski had especially close contact, one should name Robert Ball and Hobson, mathematicians, and J. J. Thomson and Rutherford.

Upon returning to Lwow, full of new ideas, he started a series of papers, simultaneously with and independently of Einstein, and in rather more detail, on the kinetic theory of gases, and developed the theory of Brownian motion.

Smoluchowski left a series of notebooks full of calculations and remarks especially noting the analogies between problems belonging to different parts of physics. The perception of analogy often precedes the discovery of more general principles. (This intuitive faculty is of transcendental value even in pure mathematics: the present writer recalls a remark by S. Banach, a great Polish mathematician: "Good mathematicians notice the analogies between theories and between methods of proof. The very great ones see the analogies between analogies.")

Smoluchowski was very adept in designing and constructing instruments. He happened to be exceedingly good at glass blowing, better than the professional mechanics at his institute. To illustrate anecdotally his liking for simple experimentation, the writer was told by Mrs.

[4] Dr. Hoerlin from Los Alamos Scientific Laboratory, a famous mountain climber himself, told me that Smoluchowski was well known for having made several first ascents in the Tatra Mountains and also in the Dolomites.

[5] The writer wants to express his gratitude to Mrs. Sofia Smoluchowski, who made available to him a wealth of material and information.

Smoluchowski how, one morning she was asked for a flat kitchen vessel, and gave her husband a salad bowl. Smoluchowski explained that he was trying to imitate a possible mechanism for the appearance of mountains on the surface of the earth. A few days later, one saw a layer of gelatin covered by a surface of mercury folding into mountain chains and ridges. This was the first step on experiments which gave rise to series of papers, on a theory of mountain formation.

The year brought a series of experiments confirming Lord Raleigh's theory that the blue of the sky was caused by the scattering of light.

As the most eminent scientist at the University of Lwow, he received many invitations to public office. For example, he was invited, in succession, to be a member of the City Council, Parliament, and Council of State. He refused all of these and did not even accept an offer of the presidency of the Scientific Society at Lwow, realizing that administrative work would take too much of his time. Nevertheless, he was an active member of several societies, the Polish Physical Society, whose first President he became. He became Dean of the Faculty of Philosophy (Arts and Sciences in our universities) during a politically difficult period for the Polish part of Austria.

There was, at that time, no textbook of theoretical physics in the Polish language. Students have edited a mimeographed collection of Smoluchowski's lectures. These he planned to use sometime in the future to prepare a large textbook of theoretical physics. As a member of the Academy of Sciences, he gave several public lectures.

In 1913, he was invited to Göttingen to give lectures on the kinetic theory of matter; the other lecturers in the series were H. G. Lorentz, M. Planck, A. Sommerfeld, and P. Debye. That year he was offered the chair of experimental physics at Cracow. Even though the University at Lwow was loath to let him leave, Smoluchowski decided to move to the University of Cracow, which was the oldest university in Poland (founded early in the 14th century), and which offered a new physical institute and the chance of being closer to the centers in the west of Europe.

World War I started the following year. Smoluchowski was evacuated with many others to Vienna during the first few months of the war. He was called into the army and worked in the military censorship.

In September, 1916, he was invited, jointly with Professor R. Zsigmondy, to give three lectures on the Brownian theory of motion and coagulation phenomena in Göttingen.

In 1917, he published his famous paper on mathematical theory of kinetics and colloidal suspensions in the *Zeitschrift fur Physikalische Chemie*. This made him one of the founders of the theory (two years after his death, W. Ostwald, the editor of *Kolloidal Zeitschrift*, published separately, in his collection, Smoluchowski's papers in this field).

In 1916 and 1917, as a Dean of the Faculty of Philosophy, he had to make trips to Vienna to get funds for the university and nominations confirmed. Also, as a member of the Organizational Committee of the newly formed School of Mines in Cracow, he made trips to Germany and Austria to institutions of this kind in Leoben, Freiburg, etc. All these activities were exhausting and only with difficulty, did he find free hours for scientific work. Mountaineering, skiing, and music became luxuries which were no longer possible, and he dreamed of being able to return to such relaxations "after the war." The horrors of war depressed him greatly, and he complained that civilization was perishing.

Political frictions at the university and among the Polish population in general affected him, and he often became pessimistic. It was in such an atmosphere that he received an invitation to a chair of theoretical physics in Vienna (after the death, on the Italian front, of one of his best friends, Professor F. Hasenöhrl). The University of Cracow kept him by making him rector (President) of the University.

Smoluchowski looked forward to his inaugural address as president and prepared it. The title was "On the Uniformity of Laws of Nature." This title was indicative of the interests of Smoluchowski during the last few years of his life. They became more general and centered in an attempted synthesis of phenomena.

He died suddenly from dysentery, after a two-week illness, on the 5th of September, 1917, at the age of 45; he had had no opportunity to learn that the Academy of Sciences at Göttingen had elected him a member.

This has rendered a brief account of external circumstances, personal contacts, and how political and national turmoils influence scientific activity. These might well be of interest to the present-day reader, accustomed to a great degree of organization and collaboration in scientific work. Smoluchowski worked as a young professor outside the great centers of scientific activity at that time. It is interesting to see how it was possible for a person of his exceptionally high ability, to get to the forefront of European thought in physics, even though the milieu in which he worked as a young professor was relatively isolated and without tradition in science. Nevertheless, it was possible to start pioneering work in a relatively new field (statistical mechanics) and get to the forefront of world science in it, once a catalyzing contact with other minds had been made. An analogous situation arose in Poland just after the end of World War I: a small group of mathematicians there, by enthusiastic work, managed to establish an impressive, new school of mathematics in fields such as set theory, topology, foundations of mathematics, etc., despite a lack of a long tradition of previous work in mathematics in this country. Such precedents might be of note if one should be surprised at the rapid rise of significant work in experimental and theoretical physics in eastern Europe at the present time.

The probability theory plays an expanding role in physics during the 40 years following the premature death of Smoluchowski. He did much to start these applications and anticipate their progress.

One of Smoluchowski's, one might say, mathematical achievements was the clarification of the role of the so-called ergodic hypothesis of Boltzmann. In a dynamical system composed of N particles, the development in time is purely deterministic. The positions and velocities are in principle calculable for all time. The whole system can be presented as one point in a space of $6N$ dimensions. This point will move as time goes on through certain of the positions within the available phase space corresponding to the given total energy of the system. That is to say, the representative point will describe a line on a surface of $6N-1$ dimensions.

Boltzmann conjectured that this line will ultimately go through every point of this surface. It was noticed, among others, by Rosenthal that this is topologically impossible. A weaker statement, however, would have been sufficient to imply the conclusions of Boltzmann, namely, the assumption that the point will get arbitrarily close to any point on the surface and will move in such a way that the time of sojourn in any sub-region of this surface will be, asymptotically, proportional to the volume of the sub-region. In this sense, the motion will present certain features of a *random* sequence of points where one would expect, if one selected positions at random rather than by deterministic equations that these points will cover the space uniformly densely.

This postulate of ergodicity is indispensable for the rigorous foundations of statistical mechanics.[6] This property still remains unproved for most of the actual dynamical systems such as for example the n-body problem. General theorems, however, were proved by J. von Neumann and G. D. Birkhoff, asserting the existence of such averages in time of sojourn in sub-regions for "almost every" point of the phase space. This is the celebrated ergodic theorem. Furthermore, one can also prove that for quite general, volume preserving, flows of phase spaces, such sojourn times are for "almost all points" indeed necessarily proportional to the volumes of the sub-regions in question.[7]

One of the points of debate which was so crucial during Smoluchowski's activity was thus settled. It shows that deterministic theories of phenomena, *in general*, lead to *randomlike* sequences of points in the representative space. Let us quote a sentence from the posthumous paper of Smoluchowski in *Naturwissenschaften* (1918).[8] "It seems to us that it is very important even for the philosopher that one can prove, even though only in a limited part of mathe-

[6] See, for example, D. ter Haar, *Elements of Statistical Mechanics* (Rinehart Publishing Corporation, New York, 1954).

[7] Cf. a paper by J. C. Oxtoby and the writer, Ann. Math. 42, 874–920 (1941).

[8] M. V. Smoluchowski, Naturwissenschaften 6, 253–263 (1918).

matical physics, that the idea of probability in the ordinary sense of a regular frequency of random effects has also an objective meaning, namely, that the idea and genesis of randomness can be made rigorously precise also if one rigorously follows the determinism; the law of large numbers comes then not as a mystical principle and not as a purely empirical fact but as a simple mathematical result of the special form which the law of causality determines in such cases." This prophetic sentence certainly is justified by the ergodic theorems proved for the first time only in 1931.

It is true that the work of mathematicians—even in the classical part of statistical mechanics, that is to say not in the quantum theoretical formulations, establishes a foundation merely for the beginning parts of thermodynamics. An analogous and rigorous treatment of the so-called H-theorem and the Boltzmann equations is not yet completed. It is interesting that Smoluchowski realized the difference in the logical structure of the theories of Maxwell, Boltzmann, on one hand, and the statistical mechanics of Gibbs. Again quoting from the same article, he says explicitly that "the difference between the two very likely consists of the fact that the former is based on certain ideas from the theory of probability, extremely plausible for these physical systems but not rigorously proved, whereas the latter avoiding these ideas, is based entirely on postulated statistical properties." This feeling for the logical structure of physical theories is extremely noteworthy when one remembers the time when these remarks were written—before the great vogue of axiomatization and the development of foundations of mathematics.

The statistical or probabilistic points of view are now used more and more widely even in the domain of classical physics. The analogies of the ergodic theorems which rigorized such treatment of systems composed of a finite number of particles came to the forefront of the work of recent years in the study of motions of continua. We have in mind statistical theories of turbulence in gases or liquids which involve infinitely many—a continuum of points. Also, the new field of magnetohydrodynamics, in the classical formulation at least, has infinitely many degrees of freedom. The equivalent of the work of Maxwell, Boltzmann, Gibbs, Smoluchowski, Poincaré, and Birkhoff, presents here a much vaster and mathematically more difficult task.

One might, at first, form the impression that the introduction of infinitely many particles or degrees of freedom is really not basic and can be attributed only to the fact that *calculus* uses such idealizations convenient mathematically for algorithms of analysis but not fundamentally necessary since gases and liquids are composed of finite assemblies of atoms. In this case, the remarks of the last few sentences would have only a methodological relevance. Unfortunately, the situation is much more difficult. The ideas of quantum theory and the role of field theory underlying the quantum theoretical description mean that one has not so far, got rid of the true continuum as a basic notion in physics. (In the beginning of the work of Maxwell *et al.*, and for the purpose of explaining the phenomena which they set themselves to do, it was sufficient to think of atoms as spheres or even as points which collided with or deflected each other, without the necessity of considering their "structure" which influences such events.)

How would Smoluchowski feel, one wonders, toward the present day formulations of quantum theory which deal *ab initio* with probabilistic concepts? One might say, of course, that the theory itself is as deterministic, or logical as ever, but the "primitive term" of it is not a point in ordinary space but rather a probability distribution. Would he feel on Einstein's side in the discussions with Bohr regarding the real meaning of determinism in physical theory? Smoluchowski discussed in his papers the necessity for continuous introduction of "hidden parameters."

In one of his papers on the uniformity of the laws of nature, he asks the question, "Why is it that the laws of nature seem as simple as they are? It is undoubtedly because we look only at small sections of phenomena or only at special features of the physical world. Now any analytic function is smooth or linear in the small." This paraphrased statement, reminiscent of Poincaré, also bears philosophical affinity to the beliefs which Einstein himself had until the end of his life. If one hopes to geometrize physics in the way that Einstein wanted to, that is to say, to describe field quantities by differentiable functions, one states a belief in the ultimate simplicity of phenomena in the small. The experience of the last two decades, with the ever more perplexing variety of forces and objects in the very small makes such hopes appear, at least for the present time, rather remote.

During Smoluchowski's lifetime, W. Ostwald defined and discussed two sharply distinct types of creative scientists: the classicists and the romantics. If one accepts such division, Smoluchowski was certainly a representative of the latter class. Together with his fine mathematical feeling and technique, he constantly searched for the transcendental relative to the then accepted schemata of physics and searched for the hidden: parameters, forces, as one would say today, operators, underlying the levels of physical description of the day.

Neglected Quadrupole*

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The usual textbook developments of the electric dipole potential are discussed and the limitations, some inherent and some arbitrary, are noted. Introducing Legendre polynomials by their generating function an alternate derivation is presented. This latter approach offers the double advantage of introducing the undergraduate to quadrupoles and higher multipoles and providing him with a powerful mathematical tool for treating electrostatic and many other problems. The importance of the quadrupole concept in modern physics is emphasized.

THIS paper is based upon a survey of some thirty texts on electricity and magnetism and on electromagnetic theory. The emphasis is on those that are supposedly on an intermediate or advanced undergraduate level. In these texts one finds a wide variety of treatments of the dipole potential. Despite the variety, the analyses are generally characterized by a bare minimum of mathematics as though a physicist should avoid mathematics at all costs. Many of the authors seem to have their eyes glued on dielectrics and exhibit no indication of knowing that there is something beyond their dielectrics. This is most unfortunate because by going just a little bit further, in some cases just by tying together the mathematical analysis given in two different parts of the book, the undergraduate student can be given two valuable additional benefits.

POTENTIAL OF THE DIPOLE

The analyses of the dipole potential found in the textbooks may be divided into three main groups. In some books, including one rather good text on theoretical physics at the graduate level, the author sets up the potential from Coulomb's law and then approximates by elementary trigonometry. The electrostatic potential φ for the system shown in Fig. 1 is given by (mks units)

$$\varphi=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}-\frac{q}{r_2}\right). \tag{1}$$

Then either by using

$$\begin{aligned} r_1 &\cong r-a\cos\theta \\ r_2 &\cong r+a\cos\theta, \end{aligned} \tag{2}$$

or some equivalent approximation, the author obtains the dipole potential

$$\varphi=\frac{1}{4\pi\epsilon_0}\cdot\frac{2aq\cos\theta}{r^2}. \tag{3}$$

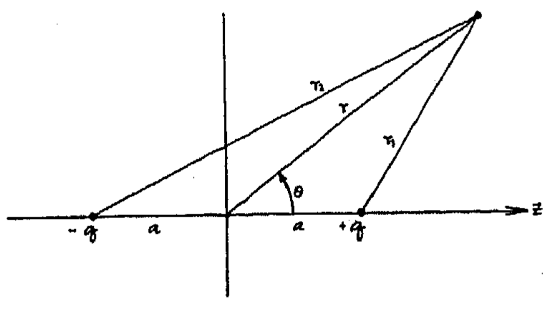

Fig. 1. Coordinate system. Charge q is at $z=a$, charge $-q$ at $z=-a$, on the z axis.

* Adapted from a paper presented at the New York meeting of the AAPT, January 1956.